The Cartan-Dieudonné Theorem
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Theorem 1. *If* $\mathbf{A}$ *is in* $\mathbf{SO}(n)$ *then there are* $\mathbf{A_1}, \ldots, \mathbf{A_n}$ *in* $\mathbf{O}(n)$ *such that* $\mathbf{A} = \mathbf{A_1} \cdots \mathbf{A_n}$*. In other words, any rotation of* n*-dimensional space can be written as a composition of at most* n *reflections.*

Proof. Let Fixed($\mathbf{A}$) be the sum of the dimensions of the subspaces fixed pointwise by $\mathbf{A}$. That is, if $S_1, \ldots, S_m$ are the subspaces of $\mathbb{R}^n$ fixed pointwise by $\mathbf{A}$ then

$$\mathrm{Fixed}(\mathbf{A}) = \sum_{i=1}^{m} \mathrm{Dim}(S_i).$$

Let $k = n - \mathrm{Fixed}(\mathbf{A})$. The integer k can be viewed as a total of all the subspaces not fixed by $\mathbf{A}$. The proof proceeds by induction on k. For the basis case, we have $k = 0$ and thus $\mathrm{Fixed}(\mathbf{A}) = n$. This means that $\mathbf{A}$ must fix all of $\mathbb{R}^n$ and so $\mathbf{A}$ must be the identity matrix. Thus we can choose $\mathbf{A_1} = \cdots = \mathbf{A_n} = \mathbf{I}$.

For the inductive step, suppose that $k > 0$ so that we have $\mathrm{Fixed}(\mathbf{A}) < n$. This last inequality implies that we can find some $\vec{x}$ in $\mathbb{R}^n$ that is not fixed by $\mathbf{A}$. Set $\vec{y} = \mathbf{A}\vec{x}$ and let $\mathbf{R}$ be the $n \times n$ matrix representing the reflection in the hyperplane with normal vector $\vec{u} = \vec{x} - \vec{y}$. Since $\vec{x}$ is not fixed by $\mathbf{A}$, the normal vector $\vec{u}$ is nonzero. The reflection matrix $\mathbf{R}$ interchanges the two vectors $\vec{x}$ and $\vec{y}$, so we have $\mathbf{RA}\vec{x} = \mathbf{R}\vec{y} = \vec{x}$. Thus $\mathbf{RA}$ fixes $\vec{x}$. It is easy to see that $\mathbf{RA}$ fixes the entire subspace S spanned by $\vec{x}$ by appealing to linearity.

Since $\mathbf{R}$ fixes pointwise the hyperplane orthogonal to $\vec{u}$, we have $\mathbf{RA}\vec{z} = \mathbf{A}\vec{z}$ for every $\vec{z}$ not in S. So $\mathbf{RA}$ behaves exactly like $\mathbf{A}$ except that it fixes the one-dimensional subspace S. We thus have $\mathrm{Fixed}(\mathbf{RA}) = \mathrm{Fixed}(\mathbf{A}) + 1$ and therefore $n - \mathrm{Fixed}(\mathbf{RA}) = n - \mathrm{Fixed}(\mathbf{A}) - 1 < n - \mathrm{Fixed}(\mathbf{A})$. By induction, we can find matrices $\mathbf{A_1}, \ldots, \mathbf{A_{n-1}}$ in $\mathbf{O}(n)$ such that $\mathbf{RA} = \mathbf{A_1} \cdots \mathbf{A_{n-1}}$. Multiplying both sides on the left by $\mathbf{R}$ and noting that $\mathbf{R}^2 = \mathbf{I}$, we see that $\mathbf{A} = \mathbf{RA_1} \cdots \mathbf{A_{n-1}}$. Each of the n factors on the right-hand side are elements of $\mathbf{O}(n)$ and so they provide the needed factorization of $\mathbf{A}$.