Representing Rotations by Quaternions

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In this article, we will assume the following reflections decomposition theorem to be true. A proof and discussion of this important theorem can be found in another article.

Theorem 1. *If* $\mathbf{A}$ *is in* $\mathbf{SO}(n)$ *then there are* $\mathbf{A_1}, \ldots, \mathbf{A_n}$ *in* $\mathbf{O}(n)$ *such that* $\mathbf{A} = \mathbf{A_1} \cdots \mathbf{A_n}$. *In other words, any rotation of* n*-dimensional space can be written as a composition of at most* n *reflections.*

We will be using the identification between vectors $\vec{x} = (x_1, x_2, x_3)$ of $\mathbb{R}^3$ and pure unit quaternions with the corresponding components, $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$.

Lemma 2. *If* $\mathbf{q}_1$ *and* $\mathbf{q}_2$ *are pure imaginary unit quaternion then* $\mathbf{q}_1\mathbf{q}_2 = \mathbf{q}_1 \times \mathbf{q}_2 - \mathbf{q}_1 \cdot \mathbf{q}_2$.

This lemma is easily seen to be true by a routine calculation. There is an immediate corollary that follows by noting that $\mathbf{q} \times \mathbf{q} = 0$ and $\mathbf{q} \cdot \mathbf{q} = |\mathbf{q}|^2 = 1$.

Corollary 3. *If* $\mathbf{q}$ *is a pure imaginary unit quaternion then* $\mathbf{q}^2 = -1$.

We can now prove an important lemma on the relationship between quaternions and reflections.

Lemma 4. *If* $\vec{n}$ *is a unit vector in* $\mathbb{R}^3$ *then the mapping* $\vec{n} \mapsto \vec{n}\vec{x}\vec{n}$ *is a reflection in the plane through the origin with normal vector* $\vec{n}$.

Proof. The mapping is linear due to the properties of quaternion multiplication. Since any vector can be decomposed in parts parallel and orthogonal to the plane orthogonal to $\vec{n}$, we can reduce the problem to considering just two different cases. The first case is when the vector $\vec{x}$ is parallel to $\vec{n}$. In this case we have $\vec{x} = s\vec{n}$ for some s in $\mathbb{R}$. Thus we have $\vec{n}\vec{x}\vec{n} = s\vec{n}\vec{n}^2 = -s\vec{n} = -x$. So $\vec{x}$ is reflected in the plane. The other case occurs when $\vec{x}$ is orthogonal to $\vec{n}$. We have $\vec{x}\vec{n} = \vec{x} \times \vec{n}$ by Lemma 2 since $\vec{x} \cdot \vec{n} = 0$ by orthogonality. Thus $\vec{x}$ and $\vec{n}$ anti-commute, that is, $\vec{x}\vec{n} = -\vec{n}\vec{x}$. It follows that $\vec{n}\vec{x}\vec{n} = -\vec{n}^2\vec{x} = \vec{x}$, so we see that the mapping fixes $\vec{x}$.

We next combine the reflections decomposition theorem with the previous lemma.

Lemma 5. *Let $\vec{m}$ and $\vec{n}$ be unit normal vectors whose corresponding planes intersect in an axis with unit direction vector $\vec{u}$ and at an angle $\theta/2$. The mapping $\vec{x} \mapsto (\vec{n}\vec{m})^* \, \vec{x} \, (\vec{n}\vec{m})$ is then a rotation around the axis $\vec{u}$ by θ.*

Proof. First we note that $(\vec{n}\vec{m})^* = \vec{m}^*\vec{n}^* = (-\vec{m})(-\vec{n}) = mn$. Using the associativity of quaternion multiplication, we thus have $(\vec{n}\vec{m})\,\vec{x}\,(\vec{n}\vec{m}) = \vec{m}\,(\vec{n}\,\vec{x}\,\vec{n})\,\vec{m}$. So by Lemma 4 we see that the mapping is a reflection in the plane orthogonal to $\vec{n}$, followed by a reflection in the plane orthogonal to $\vec{m}$. By the case $n = 3$ of Theorem 1, this is a rotation around $\vec{u}$ by the angle θ.

We now have all the lemmas we need to prove the main theorem of this article.

Theorem 6. *If $\mathbf{q} = \cos(\theta/2) + \vec{u}\sin(\theta/2)$ then the mapping $\vec{x} \mapsto \mathbf{q}^* \, \vec{x} \, \mathbf{q}$ is a rotation around the axis $\vec{u}$ by the angle θ.*

Proof. Let $\vec{n}$ be any unit normal vector orthogonal to $\vec{u}$. We can construct an orthonormal basis of $\mathbb{R}^3$ consisting of $\vec{n}$, $\vec{u}$ and $\vec{n} \times \vec{u} = \vec{n}\vec{u}$. Let $\vec{m}$ be the result of rotating $\vec{n}$ by an angle $\theta/2$ around the axis $\vec{u}$. By simple trigonometry, $\vec{m} = \vec{n}\cos(\theta/2) + \vec{n}\vec{u}\sin(\theta/2) = \vec{n}(\cos(\theta/2) + \vec{u}\sin(\theta/2))$. Multiplying both sides on the right by $\vec{n}$, we get

$$\begin{aligned}\vec{m}\vec{n} &= \vec{n}\,(\cos(\theta/2) + \vec{u}\sin(\theta/2))\,\vec{n} \\ &= -\vec{n}^2(\cos(\theta/2) + \vec{u}\sin(\theta/2)) \\ &= \cos(\theta/2) + \vec{u}\sin(\theta/2)\end{aligned}$$

where the second equality is due to the orthogonality of $\vec{u}$ and $\vec{n}$ and the third equality comes from Corollary 3. The theorem now follows from Lemma 5 if we choose $\mathbf{q} = \vec{m}\vec{n}$.